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Paper: IInd
(Differential Calculus)

Some Examples
On Basis of
Successive Derivatives
and Leibnitz's Theorem

Example: ①:- If $y = \sin mx + \cos mx$, prove that

$$y_n = m^n\left[1+(-1)^n \sin 2mx\right]^{1/2}$$

Solution:- We have $y = \sin mx + \cos mx$ — ①

nth derivative of y is given by

$$y_n = m^n \sin\left\{mx+\frac{n\pi}{2}\right\} + m^n \cos\left\{mx+\frac{n\pi}{2}\right\}$$

$$= m^n\left[\left(\sin\left\{mx+\frac{n\pi}{2}\right\}+\cos\left\{mx+\frac{n\pi}{2}\right\}\right)^2\right]^{1/2}$$

$$= m^n\left[1+2\sin\left(mx+\frac{n\pi}{2}\right)\cdot\cos\left(mx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$y_n = m^n\left[1+\sin(2mx+n\pi)\right]^{1/2}$$

$$= m^n\left[1+\sin 2mx \cos n\pi + \cos 2mx \sin n\pi\right]^{1/2}$$

$$y_n = m^n\left[1+(-1)^n \sin 2mx\right]^{1/2}$$ (Hence proved).

Example ②:- If $I_n = \frac{d^n}{dx^n}(x^n \log x)$; prove that

$$I_n = n I_{n-1} + \lfloor n-1$$

Solution:- We have $I_n = \frac{d^n}{dx^n}(x^n \log x)$ — ①

$$\Rightarrow I_n = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left\{n x^{n-1}\log x + x^n \frac{1}{x}\right\}$$

$$I_n = n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right) + \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right)$$

$$I_n = n \cdot I_{n-1} + \lfloor n-1 \quad \text{(Hence proved)}.$$

Example ③: If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$ and deduce that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

Solution: We have $y = (\sin^{-1}x)^2$ —— (i)

Differentiating w.r.t. x, we have

$$y_1 = 2(\sin^{-1}x)\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y \quad (\text{using eq}^n \text{ (i)}) \text{ —— (iii)}$$

$$\Rightarrow (1-x^2)\cdot 2y_1y_2 + (-2x)y_1^2 = 4y_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0 \text{ —— (iv)} \quad \text{(1st proved)}.$$

Using Leibnitz's theorem, we have; nth derivative of (iv),

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - y_{n+1}\cdot x - {}^nC_1 y_n \cdot 1 - 0 = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}\cdot 2y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

(2nd proved).

Exercise: ① If $\cos^{-1}(y/b) = \log\left(\frac{x}{n}\right)^n$, prove that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0.$$